Optimization of Eigenvalues When Using the Galerkin Method

P. A. A. Laura
Institute of Applied Mechanics
8111 Puerto Belgrano Naval Base,
Argentina

V. H. Cortinez
Department of Engineering
Universidad Nacional del Sur
8000 Bahia Blanca, Argentina

Introduction

Among the weighted residuals methods the Galerkin approach is one of the most popular among chemical engineers (Finlayson, 1972; Schechter, 1967).

It has been shown recently (Laura et al., 1985) that the inclusion of an undetermined parameter in the approximate coordinate function used to solve the governing differential system allows for the optimization of the fundamental eigenvalue in problems governed by the Helmholtz equation and in the case of domains of complicated boundary shape. This is due to the fact that the Galerkin method yields upper bounds for the eigenvalues. That approach is extended here in order to minimize higher eigenvalues. Two examples are given: (a) the classical Graetz problem, and (b) the determination of a concentration profile in a light-catalyzed reactor.

It is important to point out that use of the Rayleigh method and inclusion of a minimization parameter has been used extensively to determine fundamental eigenvalues in vibration and buckling problems (Schmidt, 1982; Bert, 1984).

Proposed Approach

Suppose the exact solution of an eigenvalue problem is of the form

$$y = y(x_1, x_2, \ldots, x_N) \tag{1}$$

An approximate solution y_a can then be taken of the form

$$y_a = \Sigma\, a_i\, \psi_i(x_1, x_2, \ldots, x_N, \gamma) \tag{2}$$

where the a_i's are arbitrary parameters which are determined according to the Galerkin method and γ is the minimization parameter. The functional relations ψ_i's must satisfy all the boundary conditions (Finlayson, 1972).

Substituting Eq. 2 into the governing differential equation one obtains an "error" or "residual function"

$$\epsilon = \epsilon(x_1, x_2, \ldots, x_N, a_1, a_2, \ldots, a_p, \gamma, \lambda) \tag{3}$$

where λ is the eigenvalue under investigation.

The Galerkin method requires that the error function ϵ be orthogonal with respect to each coordinate function ψ_i on the domain under consideration

$$\int_D \epsilon\, \psi_i\, dD = 0 \quad (i = 1, 2, \ldots, p) \tag{4}$$

and a linear, homogeneous system of equations in the a_i's results. From the nontriviality condition one obtains a secular determinant whose roots constitute the approximate eigenvalues of the problem λ_i. Obviously each eigenvalue λ_i is a function of the still undetermined minimization parameter γ:

$$\lambda_i = \lambda_i(\gamma) \tag{5}$$

It is required now that

$$\frac{\partial \lambda_i}{\partial \gamma} = 0 \tag{6}$$

and this minimization condition will certainly allow for the optimization of the eigenvalues. Admittedly, once the secular determinant has been generated it is convenient to perform Eq. 6 numerically.

It is important to point out that the method proposed in this note is limited to linear self-adjoint problems. Regarding the inclusion of γ in the general formulation, it appears at this point that in most cases it is convenient to use it as an exponential parameter in the approximate coordinate functions used to solve the differential system.

Applications

Consider first the Graetz problem governed by the differential system

$$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \lambda^2(1 - r^2)y = 0 \tag{7a}$$

$$y(1) = 0 \tag{7b}$$

Taking

$$y_a = \sum_{i=1}^{2} a_i(1 - r^{\gamma-1+i}) = \sum_{i=1}^{2} a_i \psi_i \tag{8}$$

and following the previously explained procedure one obtains

$$(X_2X_4 - X_6^2)\lambda^4 + (X_1X_4 + X_2X_3 - X_5X_6 - X_6X_7)\lambda^2 + (X_1X_3 - X_5X_7) = 0 \tag{9}$$

where $X_i = X_i(\gamma)$ and

$$X_1 = \int_0^1 (r\psi_1\psi_1'' + \psi_1\psi_1')dr; \quad X_2 = \int_0^1 (r - r^3)\psi_1^2 dr$$

$$X_3 = \int_0^1 (r\psi_2\psi_2'' + \psi_2\psi_2')dr; \quad X_4 = \int_0^1 (r - r^3)\psi_2^2 dr$$

$$X_5 = \int_0^1 (r\psi_1\psi_2'' + \psi_1\psi_2')dr; \quad X_6 = \int_0^1 (r - r^3)\psi_1\psi_2 dr$$

$$X_7 = \int_0^1 (r\psi_2\psi_1'' + \psi_2\psi_1')dr \tag{10}$$

As previously stated the minimization procedure is evaluated leading to the results:

$$(\gamma = 1.90) \quad \lambda_1 = 2.704$$

$$(\gamma = 1.10) \quad \lambda_2 = 7.07$$

The first eigenvalue coincides with the exact result while the second differs from the exact value by 6% ($\lambda_2 = 6.66$) (Schechter, 1967).

Consider now the determination of the concentration profile in a light-catalyzed reactor (Schechter and Wissler, 1960), which for the particular set of parameters considered by the authors in their interesting paper is expressed by the differential system

$$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \left[\lambda^2(1 - r^2) - \frac{2}{r}\right] = 0 \tag{11a}$$

$$\frac{dy}{dr}(0) = 2y(0) \tag{11b}$$

$$\frac{dy}{dr}(1) = 0 \tag{11c}$$

Taking now:

$$y_a = a_1\left(-\frac{r^\gamma}{\gamma} + r + \frac{1}{2}\right) + a_2\left(-\frac{2}{\gamma+1}r^{\gamma+1} + 2r + 1\right) \tag{12}$$

where each coordinate function satisfies the boundary conditions and applying a procedure similar to the one previously developed one obtains

$$(\gamma = 2.6) \quad \lambda_1 = 2.773$$

$$(\gamma = 1.4) \quad \lambda_2 = 6.33$$

where the first eigenvalue again coincides with the exact results (Schechter and Wissler, 1960). (The second eigenvalue is not available.)

In conclusion it appears that the present approach possesses some inherent advantages in view of its simplicity and flexibility. It may also be of considerable value in the application of the finite element method based on the Galerkin formulation.

Notation

a_i = arbitrary constant
D = domain
r = dimensionless radial variable
x_i = independent variable
X_i = see Eq. 10
Y = dependent function
γ = minimization parameter
λ = eigenvalue
ψ_i = coordinate function

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